Black rings with varying charge density

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ABSTRACT: We find the general five-dimensional, supersymmetric black ring solutions in M-theory based upon a circular ring, but with arbitrary, fluctuating charge distributions around the ring. The solutions have three arbitrary charge distribution functions, but their asymptotic charges and angular momenta only depend upon the total charges on the ring. The arbitrary density fluctuations thus represent "hair." By varying the charge distributions one can continuously change the entropy of these black rings; to our knowledge this is the first solution in which the entropy depends on classical moduli. We also show that there is a family of solutions, with two arbitrary functions, for which the horizon remains rotationally invariant, and yet the complete solution breaks rotational symmetry. If the horizon area is set to zero then one obtains families of supertube solutions. We find that our general solutions are governed by three harmonic functions that may be thought of as classical excitations of a string. The horizon area provides a natural Lorentz metric on these excitations, and the constancy of the rotational invariance of the horizon imposes a set of Virasoro constraints.

KEYWORDS: Black Holes in String Theory.

Contents

1. Introduction

There are quite a number of reasons to study and classify three-charge, BPS black ring and supertube solutions. For relativists it is interesting to understand the large variety of such solutions [1 – 6] and the large extent to which they violate black hole uniqueness. Another major driving force is the recent proposal by Mathur and collaborators that supertubes can be thought of as individual black-hole microstates and that black holes should be thought of as "statistical ensembles" of regular horizon-less microstate geometries [7 – 15].

Although string theory indicates the existence of large families of three-charge BPS solutions [16], finding these solutions has proven to be quite hard given the complexity of the underlying equations. The explicitly-known, three-charge BPS solutions are: the BMPV black hole [17, 18], supersymmetric three-charge supertubes, black rings with horizon topology $S^1 \times S^2$ in minimal supergravity [2], or eleven-dimensional supergravity [3, 4, 6, 1], superpositions of black rings [5, 6], as well as solutions obtained by other methods [11, 19, 12 – 15]. All the explicit solution are $U(1) \times U(1)$ invariant.[1] However, if one is to use these solutions to explain black hole entropy [12] or to find the extent of the violation of black hole uniqueness, it is important to find three-charge BPS supertubes, black rings and other solutions that do not have such a large symmetry.

By analyzing the Killing spinors of three-charge solutions using the Killing spinor methods developed in [20, 21], it was shown in [3] that the general problem of finding three-charge solutions that preserve the same supersymmetries as the three-charge black

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[1]Some implicit solutions with only one $U(1)$ invariance have also been constructed [5, 6].

hole can be reduced to solving *linear* equations of ordinary, Euclidean, four-dimensional electromagnetism. While these equations appear, at first sight, to be non-linear,[2] it was also pointed out in [3] that if these equations are solved in the proper sequence, then the non-linearities only appear in source terms, meaning that the problem is actually linear. To find solutions with a flat base, one first choses an arbitrary profile shape, $\vec{\mu}(\psi) \in \mathbb{R}^4$, $0 \leq \psi \leq 2\pi$, for the ring, which determines the fluxes sourced by the dipole branes. One then is free to choose three arbitrary charge density functions, $\rho_i(\psi)$, around the ring, and determine the harmonic functions sourced by these charges, which are then used to find the angular momentum and complete the solution. There are thus seven *arbitrary functions* in the general solution of [3]. While this general solution with seven arbitrary functions is not explicitly exhibited in [3], its existence is guaranteed by the existence of solutions in Euclidean electromagnetism for a given source distribution.

Even though a solution to the equations of electromagnetism may exist, the corresponding solution in M-theory is required to satisfy some more stringent physical conditions: Most particularly, there should be no closed time-like curves (CTCs). In [3] it was argued that any supertube solution, when approached sufficiently closely, should look like the straight supertube of [1], and hence be regular, and free of CTC's. However, this observation really only guarantees that there are not "local" CTC's around the ring, and also does not guarantee the absence of Dirac strings. It is therefore important to construct these solutions and examine their features explicitly.

In this paper we construct new, asymptotically flat, black ring solutions given by three of the seven arbitrary functions that govern general solutions with a flat base. The black rings and three-charge supertubes have circular dipole profiles, but have arbitrary charge densities. We will see that the charge densities feed back into the geometrical shape of the ring, causing its physical radius of curvature to vary. We also verify that the solutions have no Dirac strings and no CTC's near the ring. Although we have not analyzed whether the metric at the horizon is smooth, there is a family of our solutions, parametrized by two arbitrary functions, that has the same near-horizon geometry as the U(1) × U(1) invariant black ring, and intuitively one should expect it to be completely smooth at the horizon.[3]

Our solutions describe families of black ring solutions that have three, freely choosable charge densities around the ring, but whose asymptotic charges are the same as those of the U(1) × U(1) black ring with the same total charges. The entropy of these rings will turn out to be the integral of a certain functional of these functions, and can thus be freely varied by changing the charge densities. This freedom persists in the two parameter family that is U(1) × U(1) invariant near the horizon. Hence, our solutions are the first example of a black hole whose entropy depends on continuous parameters — the charge densities along the ring.

To find the supertube solutions one wants to set the horizon area to zero, but we consider the more general problem of setting the horizon area equal to a constant. This is interesting because it results in families of solutions that break the U(1) symmetry around

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[2]These equations were also found in work that classifies five-dimensional supergravity solutions [22, 23].

[3]The continuity issues at the horizon are rather subtle, and are being investigated by Horowitz and Reall [24].

the ring axis, but that have "round" horizons, that is, the horizon *does* have rotational invariance along the ring. Setting the horizon area to a constant imposes one functional condition, and we have three arbitrary functions in the charge densities, and so there are generally going to be families of solutions with constant horizons parametrized by two arbitrary functions.

We also find a rather simpler, and physically very interesting way to characterize our solutions. By changing coordinates, one may think of our solutions as being governed by three harmonic functions on a semi-infinite cylinder. The charge densities source the functions at the bottom of the cylinder, and the solutions are required to vanish at infinity. One may thus think of the solutions as some form of bosonic string excitations theory. The horizon radius of curvature provides a simple Lorentz metric on this bosonic string, and requiring that the radius of curvature be constant (a "round" horizon) amounts to imposing a set of Virasoro constraints. While these statements are classical, it does suggest that the charge excitations of the black ring might be quantized in terms of such a bosonic string. We will defer a more detailed investigation of these issues to a subsequent paper.

In the next section we will summarize the results of [3] concerning the system of equations that govern supertubes and black rings. In section 3 we present the solutions with varying charge densities, while section 4 discusses the solutions with round horizons, and contains a simple example. In section 5 we recast our results in terms of a classical bosonic string, and we make some final comments in section 6.

Note: After this paper was submitted to the archive, [24] appeared, in which it was claimed that black-ring solutions of minimal five-dimensional supergravity with variable charge density are not smooth. We are in the process of investigating these claims.

2. The black-ring equations

The simplest way to describe the three-charge black ring is in terms of three sets of M2 branes. We take the M2 branes to lie in the 123, 145 and 167 directions, so that they wrap tori in the spatial directions. The non-compact space-time is thus in the $1\,8\,9\,10\,11$ directions. The metric Ansatz is:

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\begin{aligned}
e^1 &= e^{-2A_1-2A_2-2A_3}\big(dx^1 + \vec{k}\cdot d\vec{y}\big), \\
e^2 &= e^{-2A_1+A_2+A_3}dx^2\,, & e^3 &= e^{-2A_1+A_2+A_3}dx^3\,, \\
e^4 &= e^{A_1-2A_2+A_3}dx^4\,, & e^5 &= e^{A_1-2A_2+A_3}dx^5\,, \\
e^6 &= e^{A_1+A_2-2A_3}dx^6\,, & e^7 &= e^{A_1+A_2-2A_3}dx^7\,, \\
e^{7+j} &= e^{A_1+A_2+A_3}dy^j\,, \quad j=1,\ldots,4\,.
\end{aligned}
\tag{2.1}
$$

The solution contains both $M2$ branes and $M5$ branes. The $M5$-branes wrap four-tori in the 4567, 2367 and 2345 directions respectively, and their last spatial dimension will define the ring profile,[4] $\vec{y} = \vec{\mu}(\psi) \in \mathbb{R}^4$, $0 \le \psi \le 2\pi$. The electric charges of the $M2$ branes are

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[4]As was pointed out in [3], the three $M5$ branes could have separate profiles, but we will not consider this possibility here since such independent profiles are almost certainly not bound states.

completely fixed in terms of the metric functions via the usual "zero-force" BPS conditions. However, the presence of the $M5$ branes is reflected in three independent Maxwell fields, $\vec{a}_{(i)}$, in the $\mathbb{R}^4$:

$$C^{(3)} = -e^1 \wedge e^2 \wedge e^3 - e^1 \wedge e^4 \wedge e^5 - e^1 \wedge e^6 \wedge e^7 + \tag{2.2}$$
$$+ 2\,(\vec{a}_{(1)} \cdot d\vec{y}) \wedge dx^2 \wedge dx^3 + 2\,(\vec{a}_{(2)} \cdot d\vec{y}) \wedge dx^4 \wedge dx^5 + 2\,(\vec{a}_{(3)} \cdot d\vec{y}) \wedge dx^6 \wedge dx^7\,.$$

It is convenient to introduce the functions, Z_j, and the Maxwell field strengths $G_{(j)}$, defined by:

$$Z_j \equiv e^{6A_j}\,, \qquad G_{(j)} \equiv d(a_{(j)})\,, \qquad j = 1, 2, 3\,. \tag{2.3}$$

The equations that define the $\frac{1}{8}$-BPS rings are then:

$$G_{(i)} = *G_{(i)}\,, \tag{2.4}$$

$$d * dZ_i = 2 \sum_{j,k} |\epsilon^{ijk}|\, G_{(j)} \wedge G_{(k)}\,, \tag{2.5}$$

$$dk + *dk = 2\,G_{(1)} Z_1 + 2\,G_{(2)} Z_2 + 2\,G_{(3)} Z_3\,, \tag{2.6}$$

where $*$ denotes the dual on $\mathbb{R}^4$ and k is the angular momentum vector appearing in e^1 in (2.1). If one solves this system in the order presented here, then the system is linear.

The general solution for a ring profile $\vec{y} = \vec{\mu}(\psi)$ can then be written in terms of the usual Green functions, as follows. First one computes:

$$\vec{b}_{(j)}(\vec{y}) \equiv \frac{q_j}{2\,\pi} \int_0^{2\pi} \frac{\vec{\mu}_j{}'(\psi)}{|\vec{y} - \vec{\mu}_j(\psi)|^2}\, d\psi\,, \tag{2.7}$$

and then sets:

$$G_{(j)} = (1 + *)\,(d\,(\vec{b}_{(j)} \cdot d\vec{y}))\,. \tag{2.8}$$

One then uses this to obtain $\vec{a}_{(j)}$. Note that the charge density in the integrand of (2.7) is constant, reflecting the fact that the number of $M5$ branes is constant along the profile. Having solved for $G_{(j)}$, one then gets the Z_i from:

$$Z_i(\vec{y}) = 1 + \int \frac{\rho_i(\vec{z}) + 2 \sum_{j,k} |\epsilon^{ijk}| * (G_j \wedge G_k)(\vec{z})}{(\vec{y} - \vec{z})^2}\, d^4z\,, \tag{2.9}$$

where the constant of integration has been set to one so that the metric has the proper asymptotics at infinity. The functions, $\rho_i(\vec{z})$, are freely choosable, but for a black ring or three-charge supertube we require the density functions to be supported on the ring. Finally, one gets the angular momentum vector, $\vec{k}$, from solving:

$$*d * dk = *2\,[(dZ_1) \wedge G_1 + (dZ_2) \wedge G_2 + (dZ_3) \wedge G_3] \equiv J\,, \tag{2.10}$$

via the Green function:

$$\vec{k}(\vec{y}) \equiv \int_{\mathbb{R}^4} \frac{\vec{J}(\vec{z})}{|\vec{y} - \vec{z}|^2}\, d^4z\,. \tag{2.11}$$

In principle one can add a homogeneous component to the solution, (2.11), that is, a vector field, $\vec{k}_0$, for which $dk_0 + *dk_0 = 0$ outside the ring. Such a solution would be sourced by an "angular momentum density," $\sigma(\psi)$, around the ring. As was noted in [3], this homogeneous solution, and the associated density, $\sigma(\psi)$, must usually be set to zero in order to avoid CTC's.

3. The solutions with varying charge densities

We now seek ring solutions with varying charge densities. We will start with a "round" $M5$-brane distribution, exactly as in [3], but we will the introduce general distributions of $M2$-brane charge around the ring. We consider a ring of $M5$ branes located at $r = 0$ and $z = R$, in the coordinates (z, ψ) and (r, ϕ) in which the metric on $\mathbb{R}^4$ is

$$d\vec{y} \cdot d\vec{y} = (dz^2 + z^2 \, d\psi^2) + (dr^2 + r^2 \, d\phi^2) \,. \tag{3.1}$$

To solve the Laplace equation and to simplify the form of the G_i it is simpler to use a better-adapted set of coordinates [2]:

$$x = -\frac{z^2 + r^2 - R^2}{\sqrt{((z-R)^2 + r^2)((z+R)^2 + r^2)}} \,, \qquad y = -\frac{z^2 + r^2 + R^2}{\sqrt{((z-R)^2 + r^2)((z+R)^2 + r^2)}} \,, \tag{3.2}$$

for which one has $-1 \le x \le 1$, $-\infty < y \le -1$, and the ring is located at $y = -\infty$. In these coordinates, the metric on $\mathbb{R}^4$ becomes:

$$ds^2_{\mathbb{R}^4} = \frac{R^2}{(x-y)^2} \left(\frac{dy^2}{y^2 - 1} + (y^2 - 1) \, d\psi^2 + \frac{dx^2}{1 - x^2} + (1 - x^2) \, d\phi^2 \right) . \tag{3.3}$$

3.1 The new solutions

Since the shapes of the dipole branes are the same as in references [3, 4, 6], the fields $\vec{a}_{(j)}$ are the same, and hence:

$$G_{(j)} = q_j \left(dx \wedge d\phi - dy \wedge d\psi \right) . \tag{3.4}$$

The new element in the solution here is to allow a general charge density around the ring. This means that Z_i contains a term of the form:

$$\lambda_i(z, r, \psi) = \int_0^{2\pi} \frac{\rho_i(\chi) \, R}{r^2 + z^2 + R^2 - 2 \, z \, R \, \cos(\psi - \chi)} \, d\chi \,. \tag{3.5}$$

where $\rho_i(\chi)$ is the linear charge density on the ring in the flat $\mathbb{R}^4$ metric. If one expands $\rho_i(\psi)$ into a Fourier series:

$$\rho_i(\psi) = a_0^i + \sum_{n=1}^{\infty} \left(a_n^i \, \cos(n \, \psi) + b_n^i \, \sin(n \, \psi) \right) , \tag{3.6}$$

then the integrals in (3.5) become elementary contour integrals and one finds

$$\lambda_i(z, r, \psi) = \frac{\pi}{R} \, (x - y) \, S_i(y, \psi) \,, \tag{3.7}$$

where y is the coordinate defined in (3.2), and

$$S_i(y, \psi) \equiv a_0^i + \sum_{n=1}^{\infty} \left(\frac{y+1}{y-1} \right)^{n/2} \left(a_n^i \, \cos(n \, \psi) + b_n^i \, \sin(n \, \psi) \right) , \tag{3.8}$$

Note that as $y \to -\infty$, $S_i(y, \psi) \to \rho_i(\psi)$. That is, as one approaches the ring, the function S_i limits to the charge density.

Thus we take:

$$Z_i = 1 + \frac{\pi}{R}\,(x-y)\,S_i(y,\psi) - \frac{4q_j\,q_k}{R^2}\,(x^2-y^2)\,, \qquad (3.9)$$

where i,j,k are all distinct, and the term proportional to $q_j\,q_k$ is a consequence of the source term in (2.9).

One of the very useful properties of the coordinates (x,y,ψ,ϕ) is that the Laplacian is separable. More precisely, one can separate variables in the function $F(x,y,\psi,\phi)$ if one acts on $(x-y)\,F$ with the Laplacian:

$$\frac{R^2}{(x-y)^3}\,\nabla^2[\,(x-y)\,F\,] = \partial_x((1-x^2)\,\partial_x F) + \partial_y((y^2-1)\,\partial_y F) +$$
$$+ (y^2-1)^{-1}\,\partial_\psi^2 F + (1-x^2)^{-1}\,\partial_\phi^2 F\,. \qquad (3.10)$$

Thus it is relatively easily to find general solutions for the Z_i in terms of orthogonal eigenfunctions. In particular, it is elementary to show directly from (2.9) that S_i must satisfy:

$$(y^2-1)\,\partial_y((y^2-1)\,\partial_y S) + \partial_\psi^2 S = 0\,, \qquad (3.11)$$

away from the ring.

It is now convenient to separate out the constant modes and normalize them to the conventions of [3], and so we set $a_0^i = \frac{Q_i}{\pi}$. It is also useful to introduce the functions:

$$\Omega_i(y,\psi) \equiv \sum_{n=1}^{\infty} \left(\frac{y+1}{y-1}\right)^{n/2} \frac{1}{n}\,(\,a_n^i\,\sin(n\,\psi) - b_n^i\,\cos(n\,\psi)\,)\,. \qquad (3.12)$$

These functions are the indefinite integrals with respect to ψ of the oscillatory part of S_i. They also satisfy the differential equation (3.11). One therefore has:

$$Z_i = 1 + \frac{Q_i}{R}\,(x-y) - \frac{4q_j\,q_k}{R^2}\,(x^2-y^2) + \frac{\pi}{R}\,(x-y)\,\partial_\psi\Omega_i(y,\psi)\,. \qquad (3.13)$$

Following [3] we also define:

$$A \equiv 2(q_1+q_2+q_3)\,, \quad B \equiv \frac{2}{R}(Q_1\,q_1 + Q_2\,q_2 + Q_3\,q_3)\,, \quad C \equiv -\frac{24q_1\,q_2\,q_3}{R^2}\,. \qquad (3.14)$$

The final step is to solve (2.6). To this end we make an Ansatz:

$$k = k_0\,dy + k_1\,d\psi + k_2\,d\phi\,. \qquad (3.15)$$

One should note that there is a gauge freedom in the definition of k since a change of coordinate $t \to t+g$ generates a gauge transformation $k \to k+dg$. The absence of a dx term in (3.15) may be viewed as a gauge choice. One then obtains the system of differential equations:

$$\partial_y k_1 - \partial_\psi k_0 - \partial_x k_2 = -\left[A + B\,(x-y) + C\,(x^2-y^2) + \frac{2\,\pi}{R}\,(x-y)\,\partial_\psi\Omega(y,\psi)\right]\,,$$
$$(1-x^2)(y^2-1)\,\partial_x k_0 - \partial_\psi k_2 - \partial_\phi k_1 = 0\,,$$
$$(y^2-1)\,(\partial_y k_2 - \partial_\phi k_0) + (1-x^2)\,\partial_x k_1 = 0\,, \qquad (3.16)$$

where

$$\Omega \equiv \sum_i q_i \, \Omega_i \,. \tag{3.17}$$

It is easy to see that a solution to this system is given by:

$$k_0 = -\frac{\pi}{R} \left[x \, (y^2 - 1)^{-1} \, \partial_\psi^2 \Omega + 2 \, y \, \Omega \right] \,,$$

$$k_1 = (y^2 - 1) \left(\frac{1}{3} \, C \, (x + y) + \frac{1}{2} \, B \right) - A \, (y + 1) + \frac{\pi}{R} \, x \, (y^2 - 1) \, \partial_\psi \partial_y \Omega \,,$$

$$k_2 = (x^2 - 1) \left(\frac{1}{3} \, C \, (x + y) + \frac{1}{2} \, B \right) - \frac{\pi}{R} \, (1 - x^2) \, \partial_\psi \Omega \,. \tag{3.18}$$

To show that this solves (3.16) one merely needs to use the fact that Ω satisfies (3.11). If $\Omega = 0$ then this is simply the solution of [3, 4, 6].

In principle one can add an arbitrary homogeneous solution of (3.16) to (3.18). However, this homogeneous solution is then fixed by requiring that there are no CTC's near the ring. This was used in [3] to fix the polynomial behavior of the k_j as functions of y. The ring is located at $y = -\infty$, and one can readily verify that Ω, $\partial_\psi^m \Omega$ and $(y^2 - 1) \, \partial_\psi \partial_y \Omega$ are all finite as $y \to -\infty$, and so it is unlikely that these new terms will generate any new CTC's near the ring, and so no further additions of homogeneous solutions should be necessary. We will see this more explicitly below.

It turns out that there is a more convenient gauge choice for k, one in which the y-component of k is zero:

$$k = \hat{k}_0 \, dx + \hat{k}_1 \, d\psi + \hat{k}_2 \, d\phi \,. \tag{3.19}$$

This may be achieved by taking $k \to k + dg$. where

$$g = -\frac{\pi}{R} \, x \, (y^2 - 1) \, \partial_y \Omega + \frac{2 \, \pi}{R} \int y \, \Omega \, dy \,. \tag{3.20}$$

One then finds:

$$\hat{k}_0 = -\frac{\pi}{R} \, (y^2 - 1) \, \partial_y \Omega \,,$$

$$\hat{k}_1 = (y^2 - 1) \left(\frac{1}{3} \, C \, (x + y) + \frac{1}{2} \, B \right) - A \, (y + 1) + \frac{2 \, \pi}{R} \int y \, \partial_\psi \Omega \, dy \,,$$

$$\hat{k}_2 = (x^2 - 1) \left(\frac{1}{3} \, C \, (x + y) + \frac{1}{2} \, B \right) - \frac{\pi}{R} \, (1 - x^2) \, \partial_\psi \Omega \,. \tag{3.21}$$

We also need to fix the function of ψ that appears as a "constant of integration" from $\int y \Omega dy$ in (3.20). This is elementary: One must not have any strings, and so this indefinite integral must vanish at spatial infinity, or as $y \to -1$. It then follows from (3.12) that this indefinite integral has a leading behavior:

$$\int y \, \Omega \, dy \sim (y + 1)^{3/2} \quad \text{as} \quad y \to -1 \,. \tag{3.22}$$

Indeed, to verify that there are no strings within our solution one must check that $\hat{k}_1$ vanishes as $z \to 0$, or $y \to -1$, and $\hat{k}_2$ vanishes as $r \to 0$, or $x \to -1$. This is evident from (3.21) and (3.22).

We now have the complete solution with arbitrary charge densities.

3.2 The near-ring limit

To examine how the metric behaves near the ring we want to take the limit as $y \to -\infty$. This structure of the metric is simpler to disentangle if we use the rotation vector (3.21), instead of (3.18). Observe that the gauge transformation (3.20) has added to k a term that diverges as $y \to -\infty$. Indeed, the leading divergence is simply:

$$\frac{\pi}{R} y^2 \left[\lim_{y \to -\infty} \partial_\psi \Omega(y, \psi) \right] \sim \frac{\pi}{R} y^2 \sum_i q_i \hat{\rho}_i(\psi). \tag{3.23}$$

where

$$\hat{\rho}_i(\psi) \equiv \rho_i(\psi) - a_0^i \tag{3.24}$$

is the oscillatory part of the charge density. Recall that the linear charge densities on the ring are $\frac{1}{\pi} \widehat{Q}_i(\psi)$, where

$$\widehat{Q}_i(\psi) \equiv Q_i + \pi \, \hat{\rho}_i(\psi). \tag{3.25}$$

Therefore by making the replacement

$$Q_i \to \widehat{Q}_i(\psi), \tag{3.26}$$

we can incorporate the leading divergence, (3.23), in k into the coefficient B defined in (3.14), and thus reduce the problem almost to that of [3].

We now consider the three-dimensional spatial part of the metric:

$$ds_3^2 = -(Z_1 Z_2 Z_3)^{-2/3} k^2$$
$$+ \frac{R^2}{(x-y)^2} (Z_1 Z_2 Z_3)^{1/3} \left((y^2 - 1) \, d\psi^2 + \frac{dx^2}{1 - x^2} + (1 - x^2) \, d\phi^2 \right) \tag{3.27}$$

as we approach the ring. Since $Z_i \sim y^2$ and $k \sim y^3$ as $y \to -\infty$ this metric potentially diverges as y^2. However one finds that the terms that are quadratic and linear in y exactly cancel, leaving a finite part:

$$ds_3^2 = \left(\frac{C^2}{9 R^2} \right)^{1/3} \left[\left(\frac{9}{C^2} \right) \widehat{M}(\psi) \, d\psi^2 + R^2 \left(d\theta^2 + \sin^2 \theta \, (d\phi + d\psi)^2 \right) \right], \tag{3.28}$$

where we have set $x = -\cos\theta$. The function, $\widehat{M}(\psi)$, is almost the obvious generalization of the parameter M in [3]:
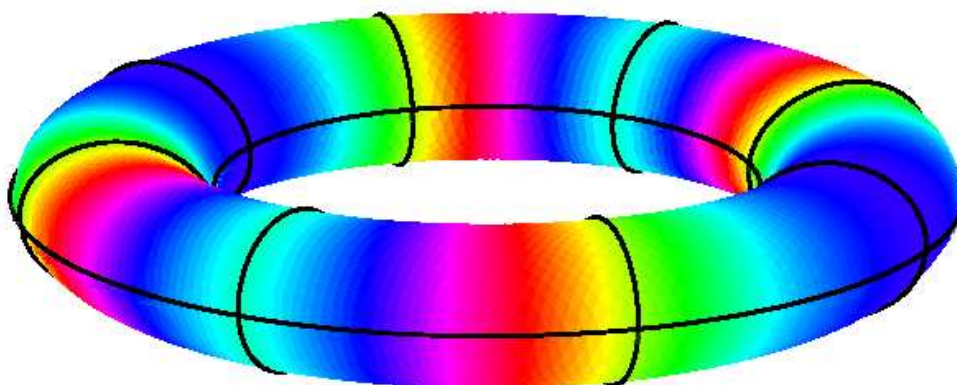
$$\widehat{M} \equiv (2q_1 q_2 \widehat{Q}_1 \widehat{Q}_2 + 2q_1 q_3 \widehat{Q}_1 \widehat{Q}_3 + 2q_2 q_3 \widehat{Q}_2 \widehat{Q}_3 - q_1^2 \widehat{Q}_1^2 - q_2^2 \widehat{Q}_2^2 - q_3^2 \widehat{Q}_3^2) + \frac{1}{3} C R^2 (A - 2\alpha(\psi)), \tag{3.29}$$

where $\widehat{Q}_i$ is defined by (3.25), and $\alpha(\psi)$ is defined via:

$$\partial_\psi \Omega = \left(\sum_i q_i \hat{\rho}_i(\psi) \right) + \frac{R}{\pi} \alpha(\psi) y^{-1} + \mathcal{O}(y^{-2}) \quad \text{as} \quad y \to -\infty. \tag{3.30}$$

In terms of the Fourier series we have

$$\alpha(\psi) = \sum_{i=1}^3 q_i \alpha_i(\psi) = \frac{\pi}{R} \sum_{i=1}^3 q_i \sum_{n=1}^\infty n \left(a_n^i \cos(n\psi) + b_n^i \sin(n\psi) \right). \tag{3.31}$$

Figure 1: The horizon of the black ring. The cross-section is really an S^2 of fixed radius, R, but is depicted here as an S^1. The variation of $g_{\psi\psi}$ along the ψ-direction in the horizon metric (3.28) is depicted as a change of color.

We therefore see that, with the exception of the $\alpha(\psi)$ term, the form of the near-ring metric is exactly that of [3], but with the replacement (3.26). This makes perfect physical sense: The functions $\widehat{Q}_i(\psi)$ are the local charge densities on the ring, and so if one approaches the ring closely then the ring should look like the infinitely long black tube [1] with charge density given by the local values.

The only unexpected modification is the appearance of the $\alpha(\psi)$ term as a shift in $A \equiv 2(q_1 + q_2 + q_3)$. The sum of the q_j is essentially the angular momentum of the black ring, and so $\alpha(\psi)$ appears to be a local shift in the angular momentum. As we will see below, none of the fluctuations like $\alpha(\psi)$ are visible from infinity. Thus, this shift must be related to the angular momentum balance of the ring locally on the ring surface. One should also note that while the ring is a perfect circle in the original $\mathbb{R}^4$ base, it is no longer circular in the complete metric. From (3.28) one can see that its radius of curvature is varying:

$$\mu(\psi) = \left|\frac{3}{C}\right| \sqrt{\widehat{M}(\psi)} \,. \tag{3.32}$$

Therefore one should expect some fluctuation in the angular momentum contribution to the horizon area.

3.3 Asymptotic charges

The region far from the ring corresponds to $x = -1, y = -1$. Indeed, in this limit:

$$x \sim -1 + \frac{2\,R^2\,r^2}{(z^2 + r^2)^2}\,, \qquad y \sim -1 - \frac{2\,R^2\,z^2}{(z^2 + r^2)^2}\,. \tag{3.33}$$

Therefore, the asymptotic charges of the solution come from the terms in the electric potentials that fall-off as $(y + 1)$. However, the $(x - y)\partial_\psi\Omega$ terms in (3.13) fall-off faster than that, and therefore the electric charge measured at infinity is exactly that of the

U(1) × U(1) invariant ring [3, 4, 6]:

$$Q_i^\infty \equiv Q_i + 4 \sum_{j,k} |\epsilon^{ijk}| \, q_j \, q_k \tag{3.34}$$

One can convert these charges to number densities of branes using:

$$Q_i = \frac{\bar{N}_i l_p^6}{2L^4 R}, \qquad q_i = \frac{n_i l_p^3}{4L^2}, \tag{3.35}$$

where L is the length of the two-tori. The asymptotic charges, N_i, of the solution are then the sum of the charges on the black ring $\bar{N}_i$, and the charges dissolved in fluxes:

$$N_1 = \bar{N}_1 + n_2 n_3 \,, \quad N_2 = \bar{N}_2 + n_1 n_3 \,, \quad N_3 = \bar{N}_3 + n_1 n_2 \,. \tag{3.36}$$

The angular momenta also behave in exactly the same way: They can be read off from the $dt\,d\psi$ and $dt\,d\phi$ terms in the metric that vanish as $(1+x)^a(y+1)^{(1-a)}$, for some a, or inversely as the square of the distance. From (3.21) and (3.22) one sees that the terms involving Ω do not contribute; hence the angular momenta are also the same as for the uniformly charged ring:

$$J_1 = J^T + \frac{1}{2}\left(\sum_{i=1}^3 n_i N_i - n_1 n_2 n_3 \right), \qquad J_2 = -\frac{1}{2}\left(\sum_{i=1}^3 n_i N_i - n_1 n_2 n_3 \right), \tag{3.37}$$

where J^T is the angular momentum carried by the ring:

$$J^T = \frac{R^2 L^4}{l_p^6} \left(n_1 + n_2 + n_3 \right). \tag{3.38}$$

Thus the varying charge densities are undetectable in the asymptotic charges, and so the fluctuating charge densities, $\hat{\rho}_i(\psi)$, truly represent an infinite amount of "hair" on the black ring.

4. Some properties of the solutions

We have exhibited a family of solutions with three arbitrary fluctuating charge densities, $\hat{\rho}_i(\psi)$, and the asymptotic charges of the solution are completely independent of these fluctuating densities. On the other hand, the horizon geometry, and its area, do depend non-trivially upon these functions. In particular, the function $\widehat{M}(\psi)$ is a quadratic functional of these densities. Indeed, the $\widehat{Q}_i$ (which appear quadratically in (3.29)) are linear functions of the $\hat{\rho}_i$, and the function $\alpha(\psi)$ is also a linear functional of $\hat{\rho}_i(\psi)$, as can be seen from (3.31).

Supertubes are, of course, obtained by setting $\widehat{M}(\psi)$ identically equal to zero. Since this involves one functional constraint on three arbitrary functions, our solutions will contain a family of supertubes parametrized by two independent functions. Before examining this more closely, it is interesting to broaden the issue a little, and discuss families of solutions for which $\widehat{M}(\psi)$ is actually a constant, and not necessarily zero. These solutions are physically

very interesting because, even though the complete solution is ψ-dependent and breaks ψ-translation symmetry, the *horizon* does have ψ-translation symmetry if $\widehat{M}(\psi)$ is constant. Thus the geometry near the horizon is identical to that of the round ring, and the solutions provide a very graphic example of black hole hair, as they have two functions-worth of hair that lives on a completely round ring.

As an example, consider charge density modes that only have a finite number of Fourier terms - that is, the series (3.6) ends at $n = K$. Then one can simply substitute (3.6) and (3.31) into $\widehat{M}(\psi)$. The result is then a Fourier series in $\sin(n\psi)$ and $\cos(n\psi)$ for $n \leq 2K$. Setting all but the constant term to zero will impose $4K$ constraints on the $6K$ variables a_n^i, b_n^i, $n = 1, \ldots, K$, $i = 1, 2, 3$. Thus we get $2K$ free parameters in general. One should also note that in simplifying $\widehat{M}(\psi)$, one will use the identities $\cos^2(n\psi) = \frac{1}{2}(1 + \cos(2n\psi))$, $\sin^2(n\psi) = \frac{1}{2}(1 - \cos(2n\psi))$, and this will shift the constant terms in $\widehat{M}(\psi)$ by $\frac{1}{2}(a_n^i)^2$ and $\frac{1}{2}(b_n^i)^2$. Thus the fluctuating modes *will* contribute to the horizon area.

It is elementary to implement the foregoing analysis for a single Fourier mode. Take $a_n^i = \frac{R}{\pi} c_i$ and $b_n^i = \frac{R}{\pi} d_i$ for some n, and set all the other modes to zero. Then

$$\widehat{Q}_i = Q_i + R\left(c_i \cos(n\psi) + d_i \sin(n\psi)\right). \tag{4.1}$$

and

$$\alpha(\psi) = \sum_{i=1}^{3} q_i \, n \left(c_i \cos(n\psi) + d_i \sin(n\psi)\right). \tag{4.2}$$

Let $\vec{c}$ and $\vec{d}$ be the vectors whose components are c_i and d_i, and define:

$$\vec{Q} = (Q_1, Q_2, Q_3), \qquad \vec{q} = (q_1, q_2, q_3), \qquad \vec{q}_c = \frac{1}{R}(q_2 q_3, q_1 q_3, q_1 q_2). \tag{4.3}$$

Introduce the matrix:

$$\mathcal{M} \equiv \begin{pmatrix} -q_1^2 & q_1 q_2 & q_1 q_3 \\ q_1 q_2 & -q_2^2 & q_2 q_3 \\ q_1 q_3 & q_2 q_3 & -q_3^2 \end{pmatrix}, \tag{4.4}$$

then the condition that $\widehat{M}(\psi)$ is independent of ψ yields the equations:

$$(\vec{Q} + 8\, n\, \vec{q}_c)^T \cdot \mathcal{M} \cdot \vec{c} = 0, \qquad (\vec{Q} + 8\, n\, \vec{q}_c)^T \cdot \mathcal{M} \cdot \vec{d} = 0,$$
$$(\vec{c})^T \cdot \mathcal{M} \cdot \vec{d} = 0, \qquad (\vec{c})^T \cdot \mathcal{M} \cdot \vec{c} = (\vec{d})^T \cdot \mathcal{M} \cdot \vec{d}. \tag{4.5}$$

One also finds that the value of $\widehat{M}$ is shifted from its value when $\hat{\rho}_i = 0$ by an amount:

$$\Delta\widehat{M} = \frac{1}{2} R^2 \left[(\vec{c})^T \cdot \mathcal{M} \cdot \vec{c} + (\vec{d})^T \cdot \mathcal{M} \cdot \vec{d}\right]. \tag{4.6}$$

Therefore, in the (indefinite) metric defined by $\mathcal{M}$, the vectors $(\vec{Q} + 8\, n\, \vec{q}_c)$, $\vec{c}$ and $\vec{d}$ must all be mutually orthogonal, and $\vec{c}$ and $\vec{d}$ must have the same norm. There are thus two parameters: The norm of $\vec{c}$ and the freedom to make a rotation in the $(\vec{c}, \vec{d})$ plane. The shift in the horizon parameter $\widehat{M}$ is proportional to the norm-squared of $\vec{c}$.

If one has $Q_i = Q$ and $q_i = q$, $i = 1, 2, 3$, then, in the standard inner product in $\mathbb{R}^3$ the vectors $\vec{c}$, $\vec{d}$ and $(1, 1, 1)$ must be mutually orthogonal, with $\vec{c} \cdot \vec{c} = \vec{d} \cdot \vec{d}$, and then

$$\widehat{M} = 3\, q^2\, (Q^2 - 16\, q^2) - q^2\, R^2\left[\vec{c} \cdot \vec{c} + \vec{d} \cdot \vec{d}\right]. \tag{4.7}$$

5. A conformal field theory on the ring?

The calculation outlined in the last section actually amounts to solving the classical Virasoro constraints for two-dimensional scalar fields whose target space is a (2+1)-dimensional, Lorentzian space. This suggests that one might understand the "hair" defined by the charged modes in terms of a conformal field theory.

Introduce three scalar fields:

$$X^j(w, \bar{w}) = \frac{1}{2}\, \alpha_0^j \, \log(w\,\bar{w}) + \sum_{n \neq 0} \frac{1}{n} \left[\, \alpha_n^j \, w^n + \tilde{\alpha}_n^j \, \bar{w}^n \,\right], \tag{5.1}$$

where, for the moment, the sum over n runs from $-\infty$ to ∞. Let $w = e^{-\tau + i\psi}$, and observe that the $X^j(w, \bar{w})$ satisfy the harmonic equation:

$$(\partial_\tau^2 + \partial_\psi^2)\, X^j = 0. \tag{5.2}$$

To relate this to the results of the previous section, note that if one makes the change of variable:

$$y = -\frac{\cosh \tau}{\sinh \tau}, \tag{5.3}$$

then equation (3.11) becomes exactly the harmonic equation (5.2). We now rescale the functions S_j of the previous section, and set:

$$\pi\, q_j\, S_j = \partial_\tau X^j, \tag{5.4}$$

with no sum on j. In making this identification we must set $\alpha_n^j = \tilde{\alpha}_n^j = 0$ for $n < 0$ and impose the reality condition $\tilde{\alpha}_n^j = (\alpha_n^j)^*$. Setting the negative modes to zero is required in order to make the solution regular at $\tau = \infty$, or $y = -1$, and in particular, this means that the solution is regular at spatial infinity in the original metric (3.1).

Introduce the matrix:

$$\mathcal{P} \equiv \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \tag{5.5}$$

and define:

$$T \equiv -(\partial_\tau \vec{X})^T \cdot \mathcal{P} \cdot (\partial_\tau \vec{X}) - \frac{16\, q_1\, q_2\, q_3}{R} \left(\sum_{j=1}^3 (\partial_\tau^2 X^j) \right) - 16\, q_1\, q_2\, q_3\, (q_1 + q_2 + q_3). \tag{5.6}$$

Then one can easily check that:

$$\widehat{M}(\psi) = \lim_{\tau \to 0} T. \tag{5.7}$$

Observe that $\mathcal{P}$ has eigenvalues $-1, +1, +1$ and so defines a Lorentzian metric on $\mathbb{R}^3$. Moreover, T represents the kinetic part of an (indefinite) Hamiltonian for a scalar field theory with a "charge at infinity." That is, T contains a kinetic term for the scalar fields, but does not contain the "elastic terms," $(\partial_\sigma X^j)^2$. However, if one only considers solutions

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with a round horizon, and hence constant $\widehat{M}(\psi)$, then in terms of T, this condition amounts to the Virasoro constraints:

$$L_n \equiv \frac{1}{2\pi} \int_0^{2\pi} e^{in\psi} T(\tau, \psi)|_{\tau=0}\, d\psi = 0, \qquad n > 0. \tag{5.8}$$

This Virasoro condition reduces the degrees of freedom by one bosonic function. Note that if this were a string theory, then T would contain energy terms of the form, $(\partial_\sigma X^j)^2$, and the theory would possess conformal/reparametrization invariance. The latter would reduce the degrees of freedom by one additional function, leaving only one transverse set of string modes.[5] Here we do not appear to have this additional harmonic reparametrization invariance, $\tau \to \tilde{\tau}(\tau, \psi)$ and $\psi \to \tilde{\psi}(\tau, \psi)$, because the other physical fields, like the metric (3.3), are not invariant under such reparametrizations. Thus, one should probably think of this as free bosons on a space with indefinite signature, and there are still two freely choosable arbitrary functions in the solution after one solves (5.8).

For $\widehat{M}(\psi) = 0$, the free bosons described above will encode a lot of zero entropy configurations that one might use to explain part of the entropy of black holes and black rings. One of the main problems with counting three-charge supergravity configurations is quantizing them. If one can reinterpret the foregoing classical free bosons in terms of a conformal field theory then this may well provide a natural way to quantize geometries, similar to that in [25].

6. Final comments

We have constructed a huge family of supersymmetric black ring solutions with arbitrarily varying charge densities around the ring. These solutions do not appear to have any CTC's, and indeed bear out the intuition that if one approaches the ring closely then it will look like an infinite black cylinder whose charge density is set by its local value. In particular, we found that the near-ring metric could essentially be obtained from that of [3, 4, 6] by replacing the constant charge densities by the varying density functions, $\widehat{Q}_i(\psi)$. The only correction to this prescription involves an angularly dependent shift in the contribution to the horizon area coming from the angular momentum of the ring. While the ring has a lot of structure coming from the fluctuating charge densities, $\hat{\rho}_i(\psi)$, none of this structure contributes to the asymptotic charges. The charges of our new black rings are exactly those of the U(1) × U(1) invariant black ring, and so the functions $\hat{\rho}_i(\psi)$ do indeed represent non-trivial "hair."

Even if the shape of the dipole branes forming the ring is round in the flat $\mathbb{R}^4$ base, the varying charge densities feed back into the metric, so that the radius of curvature of the ring fluctuates. Indeed, the ring radius, $\mu(\psi) \sim \sqrt{\widehat{M}(\psi)}$, is quadratic in the fluctuating densities, $\widehat{Q}_i(\psi)$. Thus a fluctuation in $\widehat{Q}_i(\psi)$ that has n nodes could give rise to a fluctuation in the ring radius that has $2n$ nodes (depending upon whether the quadratic on linear terms dominate). By tuning only one parameter we can obtain solutions that interpolate between black ring and supertube as one goes along the ring. While our analysis

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[5]This is typically used in light-cone gauge to set $X^+ \sim p^+ \tau$.

does not detect any obvious problem with such solutions (apart from the well-known null orbifold), one might reasonably expect a more detailed analysis of the geometry to reveal some singular behavior near the transition points.

Probably the most physically relevant subset of our solutions has constant $\widehat{M}(\psi)$, and is parametrized by two arbitrary functions. If $\widehat{M} = 0$ these solutions are three-charge supertubes. If $\widehat{M} > 0$ then these solutions describe black rings whose near-horizon geometry is the same as that of the $U(1) \times U(1)$ invariant ring, which means that at least the rings with constant $\widehat{M}$ should have regular horizon structure. We have also found that these solutions can be described by a set of three free bosons satisfying a Virasoro constraint and with a target space of signature (2,1).

Solutions with constant $\widehat{M}$ are also a good starting point for constructing non-BPS three-charge black rings (generalizing the rings found in [26 – 29]). A variable $\widehat{M}$ non-extremal ring would almost certainly be time dependent: The rotating lumps would radiate energy away as gravitational waves, and the solution would evolve to a rotating ring in which he horizon was "round." By the same token, one might also wonder if non-BPS solutions with varying charge densities, but constant horizon radius, would be unstable through electromagnetic radiation, and thus decay into states with uniform charge densities. This would be very interesting to pursue further.

In [3] it was shown that the most general black-ring solution is given by seven arbitrary functions: the four independent profile functions, $\vec{y}(\psi)$, and the three charge densities. It would be very interesting to find this general solution explicitly, and to see if the configuration with a round horizon is similarly described by free bosons with a seven-dimensional target space. Understanding this sigma model might be the key to quantizing the shape and density profiles that give three-charge geometries. The entropy of our rings is an integral of a functional of the charge densities and can be freely varied by changing these densities. This freedom persists in the two parameter family of solutions that are $U(1) \times U(1)$ invariant near the horizon. Even if the densities and the corresponding entropy are classically continuous, at a quantum level one expects this continuity to be lost, as happens for the two charge supertube [30], where the discrete spacing between different shapes and density profiles accounts for the entropy of the tubes [31 – 33]. Quantizing the space of supertubes in terms of a CFT might open the door to computing the entropy in three-charge supertubes, and finding if this matches the black ring or black hole entropy.

Last, but not least, the $U(1) \times U(1)$ invariant rings have a very simple description in terms of the D1-D5 CFT [34]. It would be nice to extend this description to the new black ring solutions we have constructed here, and to match their entropy to that of a sector of the D1-D5 CFT.

Acknowledgments

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